Multiple-Valued Decomposition of Generalized Boolean Functions and the Complexity of Programmable Logic Arrays

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Abstract—Generalized Boolean functions are shown to be useful for the design of programmable logic arrays (PLA's), and the complexity of three types of PLA's is obtained by the theory of multiple-valued decomposition. A two-level PLA consists of an AND array and an OR array, and they are cascaded to perform a two-level AND-OR circuit. A PLA with decoders consists of decoders, an AND array, and an OR array. A three-level PLA consists of a $D$ array, an AND array, and an OR array, and they are cascaded to perform a three-level OR-AND-OR circuit. It is shown that a generalized Boolean function $f(x_1, x_2, \ldots, x_r)$, where $B = \{0, 1\}$, is represented by a generalized Boolean expression of $2^n$-valued variables $X_i$; and $f$ can be directly realized by a PLA with decoders or a three-level PLA. To realize a function of $n$ variables, the following sizes are shown to be sufficient: for a two-level PLA, $(n + \frac{1}{2}2^n$; for a PLA with two-bit decoders, $(\frac{3n}{2} + \frac{1}{2}2^n$; for a three-level PLA, $2^n + (3n + 1)\sqrt{2^n + 2n}. Especially in the case of PLA with two-bit decoders, the following sizes are shown to be necessary and sufficient: for an arbitrary symmetric function, $(\frac{3n}{2} + \frac{1}{2})\sqrt{2^n}$; and for a parity function, $(n + \frac{1}{2})\sqrt{2^n}$.

Index Terms—Complexity of logic circuits, functional decomposition, multiple-valued logic, programmable logic array, symmetric function.

I. INTRODUCTION

PROGRAMMABLE logic arrays (PLA's) are known as an approach to implement logic circuits having low production potential [1] [4]. In this paper the complexity of three types of PLA's is discussed: a two-level PLA, a PLA with decoders, and a three-level PLA. The first type of PLA, a two-level PLA, is shown in Fig. 1. It consists of an AND array and an OR array. For example, the two-level PLA in Fig. 2 realizes the function of Table I. The AND array generates products of input variables, and the OR array generates sums of products. This PLA corresponds to a two-level AND-OR circuit. We define the size of this PLA as $C(n) = (2n + m)W$. The second type of PLA, a PLA with decoders, is shown in Fig. 3. Each decoder generates all the maxterms of its input variables. For example, the PLA in Fig. 4 realizes the function of Table I. We define the size of a PLA with decoders as $(H + m)W$. The two-level PLA can be considered as a special case of a PLA of this type, i.e., a PLA with one-bit decoders. The third type of PLA, a three-level PLA, is shown in Fig. 5. It is a PLA with a programmable input decoders. The $D$ array generates the sum of input variables. This PLA corresponds to a three-level OR-AND-OR circuit. For example, the PLA in Fig. 6 realizes the function of Table I. We define the size of this PLA as $C(n) = (2n + W)H + Wm$.

In Section II we introduce generalized Boolean functions and their expressions. They are useful for the design of PLA's with decoders or three-level PLA's. Generalized Boolean functions can be directly realized by PLA's with decoders or three-level PLA's.

In Section III we introduce the theory of multiple-valued decomposition of generalized Boolean functions. We use this theory to obtain the complexity of PLA's in Sections IV and V. Section III is somewhat involved and the reader may skip it for the first reading.

In Section IV, first we show that AND arrays of PLA's with decoders can be minimized by using generalized Boolean functions, then we obtain the size of PLA's with decoders for various classes of functions.

Manuscript received April 25, 1980; revised January 14, 1981 and February 18, 1981. This work was supported in part by the Ministry of Education of Japan under Grants 375187 and 475275.

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TABLE I

<table>
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<tr>
<th>$x_1$</th>
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In Section V, first we show that AND arrays of three-level PLA's can be minimized by using generalized Boolean functions, then we obtain the size of three-level PLA's for various classes of functions.

Table II shows the sizes of PLA's for various classes of functions, which are obtained by using the theory of Section III. Table III shows the average size of PLA's for randomly generated functions, which are obtained by a computer experiment.

II. GENERALIZED BOOLEAN FUNCTION

In this section we introduce generalized Boolean functions and their expressions.

An ordinary function $f(x_1, x_2, \ldots, x_n): B \times B \times \cdots \times B \rightarrow B$, $B = \{0, 1\}$, can be represented by a Boolean expression of two-valued variables $x_i (i = 1, 2, \ldots, n)$, whereas the generalized Boolean function $f(X_1, X_2, \ldots, X_r): B^n \times B^n \times \cdots \times B^n \rightarrow B$, can be represented by a generalized Boolean expression of $2^n$-valued variables $X_i (i = 1, 2, \ldots, r)$.

**Definition 1**: Let $X = \{x_1, x_2, \ldots, x_n\}$ be a variable in $B^n$, where $B = \{0, 1\}$. The set of the variables in $X$ is denoted by $\{X\}$. $(X_1, X_2, \ldots, X_r)$ is said to be a partition of $X$ iff $\{X_1\} \cup \{X_2\} \cup \cdots \cup \{X_r\} = \{X\}$, $\{X_i\} \cap \{X_j\} = \emptyset$ ($i \neq j$), and $\{X_i\} \neq \emptyset$ for all $i$ and $j$. The number of the variables in $\{X_i\}$ is denoted by $d(X_i)$. 

Fig. 4. PLA with two-bit decoders for Table I.

Fig. 5. Three-level PLA.

Fig. 6. Three-level PLA for Table I.
TABLE II

<table>
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<tr>
<th>n</th>
<th>Particle Function</th>
<th>Arbitrary Function</th>
<th>Symmetric Function</th>
<th>Particle Function</th>
<th>Arbitrary Function</th>
<th>Symmetric Function</th>
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**Definition 2:** Let \( a = (a_1, a_2, \ldots, a_n) \) be a constant in \( B^n \). A symbol \( X^a \) denotes a mapping \( X^a : B^n \to B \), such that \( X^a = 0 \) if \( X \neq a \) and \( X^a = 1 \) if \( X = a \). Let \( S \subseteq B^n \). A symbol \( X^S \) denotes the function such that

\[
X^S = \bigvee_{a \in S} X^a.
\]

**Example 1:** Let \( X = (x_1, x_2) \), \( a = (0, 1) \) and \( b = (1, 0) \).

\[
X^a = \begin{cases} 
1 & \text{if } X = (0, 0), \\
0 & \text{if } X \neq (0, 0),
\end{cases} \quad \text{and} \quad X^{a,b} = \begin{cases} 
1 & \text{if } X = (0, 1) \text{ or } X = (1, 0), \\
0 & \text{if } X = (0, 0) \text{ or } X = (1, 1).
\end{cases}
\]

For simplicity, \( X^{(0,1),(1,0)} \) is denoted by \( X^{01,10} \).

**Lemma 1:** Let \( d(X) = n \) and \( S_1, S_2 \subseteq B^n \).

\[
X^{S_1} \cdot X^{S_2} = X^{S_1 \cap S_2},
\]

\[X^{S_1} \lor X^{S_2} = X^{S_1 \cup S_2}, \quad X^{\emptyset} = 1, \quad X^0 = 0.
\]

Let \( B = [X^S : S \subseteq B^n] \). The system (\( B, 0, 1, \lor, \cdot, \neg \)) is a \( 2^n \)-element Boolean algebra.

**Example 2:** Let \( X = (x_1, x_2), S_1 = [00, 10], \) and \( S_2 = [10, 11] \).

\[
X^{[10,01]} = X^{10,01}, \quad X^{[10,01]} = X^{10,01}, \quad X^{[10,01]} = X^{10,01}, \quad X^{[10,01]} = X^{10,01}, \quad X = (x_1, x_2), \quad X^0 = 0.
\]

**Definition 3:** \( X^S \) is said to be a **literal**. A product of distinct literals is said to be a **term**. A sum of terms is said to be a **sum-of-products expression**. The number of terms in a sum-of-products expression \( P \) is denoted by \( t(P) \). \( P \) is said to be minimal if there is no expression \( Q \) such that \( t(Q) < t(P) \) and \( Q \) denotes the same function as \( P \).

**Theorem 1:** Let \( X = (x_1, x_2, \ldots, x_n) \) be a partition of \( X \). An arbitrary function \( f(X) \) can be represented in the form

\[
\bigvee_{(a_1, a_2, \ldots, a_n)} f(a_1, a_2, \ldots, a_n) \cdot X^{a_1} \cdot X^{a_2} \cdot \ldots \cdot X^{a_n}
\]

where \( a_i \in B^n \) and \( n_i = d(X_i) \).

**Example 3:** 1) \( X_1 = (x_1), X_2 = (x_2), X_3 = (x_3), \) and \( X_4 \) is a trivial case of generalized Boolean expression and essentially the same as the ordinary one. However, 2) is a nontrivial case of generalized Boolean function. Note that the number of terms in (5) is smaller than that of (4).

III. MULTIPLE-VALUED DECOMPOSITION OF GENERALIZED BOOLEAN FUNCTIONS

In Section III-A we introduce multiple-valued decomposition of generalized Boolean functions. In Section III-B we obtain the number of terms in a generalized Boolean expression for various classes of functions.

A. Decomposition Theory

**Definition 4:** Let \( X = (x_1, x_2, \ldots, x_n) \) be a partition of \( X \), and \( f(X) \) be a function such that

\[
f(X_1, X_2, \ldots, X_n) = X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n}
\]
For \( a, b \in B^n \), define a relation \( \equiv \) such that

\[ a \equiv b \iff f(X|X_i = a) = f(X|X_i = b) \]

where \( f(X|X_i = a) \) denotes \( f(X_1, X_2, \ldots, X_{i-1}, a, X_{i+1}, \ldots, X_r) \). Obviously, the relation \( \equiv \) is an equivalence relation. Let \( \Pi_i = (L_0, L_1, \ldots, L_{k_i - 1}) \) be a partition of \( B^n \) induced by the equivalence relation \( \equiv \). A function \( \psi_i : B^n \rightarrow M_i; M_i = \{0, 1, \ldots, k_i - 1\} \) such that \( \psi_i(a) = j \iff a \in L_j \) is called a partition function of \( B^n \), where \( 1 \leq k_i \leq 2^n \).

**Example 5:** Consider a six variable function

\[ f(X) = (x_1 \vee x_2) \cdot (x_3 \oplus x_4) \cdot (x_5 \vee x_6) \]

\[ \vee (x_1 \vee x_2) \cdot (x_3 \oplus x_4) \cdot x_5 \cdot (x_1 \oplus x_2) \cdot (x_5 \oplus x_6). \]

Let \( (X_1, X_2, X_3) \) be a partition of \( X \), where \( X_1 = (x_1, x_3, x_2) \), \( X_2 = (x_3, x_4) \), and \( X_3 = (x_5, x_6) \). The function \( f(X) \) has the properties that

\[ f(X|X_1 = (01)) = f(X|X_2 = (10)) = f(X|X_2 = (00)) \]

\[ = f(X|X_2 = (11)) \]

\[ = f(X|X_2 = (01)) \]

\[ = f(X|X_2 = (10)). \]

Partitions on \( B^n \) induced by the equivalence relations \( \equiv \), \( i = 1, 2, 3 \) are \( \Pi_1 = \{(00), (01), (11)\} \), \( \Pi_2 = \{(00), (10), (01)\} \), and \( \Pi_3 = \{(00), (10), (11)\} \), respectively. The partition functions of \( B^n \) are shown in Table IV.

**Lemma 2:** Let \( (X_i, X_2, \ldots, X_r) \) be a partition of \( X, d(X_1) = n_i \), and let \( \psi_1 \) be a partition function of \( B^n \). There exists a multiple-valued input two-valued output function \( g : M_1 \times M_2 \times \cdots \times M_r \rightarrow B \) such that \( f(X_1, X_2, \ldots, X_r) = g(\psi_1(X_1), \psi_2(X_2), \ldots, \psi_r(X_r)) \), where \( 1 \leq |M_i| = k_i \leq 2^n \).

**Proof:** For each \( b_1 \in M_1 \), there exists \( a_1 \) such that \( \psi_1(a_1) = b_1 \). Let \( g(b_1, a_2, \ldots, a_r) = f(a_1, a_2, \ldots, a_r) \). It is easy to show that this function satisfies the condition of the theorem.

**Example 6:** Consider the function of Example 5. By Lemma 2 \( f(X) \) can be represented as \( f(X_1, X_2, X_3) = g(\psi_1(X_1), \psi_2(X_2), \psi_3(X_3)) \), where \( g(Y_1, Y_2, Y_3) \) is shown in Table V.

**Definition 5:** Let \( M = \{0, 1, \cdots, k - 1\} \), \( t \in M \), and \( Y^t : M \rightarrow B \) be a function such that \( Y^t = 0 \) if \( Y^t \neq t \) and \( Y^t = 1 \) if \( Y^t = t \). Let \( T \subseteq M \), \( Y^T \) denotes a function such that \( Y^T = \bigvee_{t \in T} Y^t \).

**Lemma 3:** Let \( T_1, T_2 \subseteq M = I \).

\[ Y^{T_1} \cdot Y^{T_2} = Y^{T_1 \cap T_2}, Y^{T_1} \vee Y^{T_2} = Y^{T_1 \cup T_2}, \]

\[ Y^{T_1} \cdot Y^{T_1} = Y^T, Y^T = 1, \text{ and } Y^{\phi} = 0. \]

Let \( B = \{Y^T : T \subseteq M\} \). The system \((B, 0, 1, \vee, \cdot, =)\) is a Boolean algebra.

**Lemma 4:** A multiple-valued input two-valued output function \( g : M_1 \times M_2 \times \cdots \times M_r \rightarrow B \) can be represented in the form

\[ g(Y_1, Y_2, \cdots, Y_r) = \bigvee_{(t_1, t_2, \cdots, t_r)} Y_1^{t_1} \cdot Y_2^{t_2} \cdots Y_r^{t_r} \]  

(6)

**Table IV**

| \( a \_i \) | \( e(X_1) \_e(X_2) \_e(X_3) \) |
|---|---|---|
| 00 | 0 | 0 |
| 01 | 1 | 1 |
| 10 | 1 | 1 |
| 11 | 2 | 0 |

**Table V**

| \( Y_1 \_Y_2 \_Y_3 \_R \) |
|---|---|---|---|
| 00 | 0 | 0 | 0 |
| 01 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 |
| 11 | 2 | 0 | 0 |

or in a form

\[ g(Y_1, Y_2, \ldots, Y_r) = \bigvee_{(t_1, t_2, \ldots, t_r)} Y_1^{t_1} \cdot Y_2^{t_2} \cdots Y_r^{t_r} \]  

(7)

where \( i_j \in M_i, Y^T = \bigvee_{i_j \in T_j} Y^{i_j} \) and \( M_i = \{0, 1, \cdots, k_i - 1\} \).

**Proof:** By Definition 5 it is easy to show that (6) holds. By Lemma 3 and (6) we have (7).

**Example 7:** The function \( g \) of Table V can be represented in the form

\[ g(Y_1, Y_2, Y_3) = Y_1^{0/1} \cdot Y_2^{1/2} \cdot Y_3^{2/3} \]  

or in a form

\[ g(Y_1, Y_2, Y_3) = Y_1^{0/1} \cdot Y_2^{1/2} \cdot Y_3^{2/3} \]  

(8)

**B. Number of Terms**

**Lemma 5:** Let \( f, \psi_1, \psi_2, \) and \( g \) be the functions of Lemma 2. A literal \( Y_1^{t_1} \) of the expression (7) corresponds to the literal \( X_1^{S_1} \) of (2), where \( S_1 = \psi_1^{-1}(T_1) \). And a term \( Y_1^{t_1} \cdot Y_1^{t_2} \cdot \cdots \cdot Y_1^{t_r} \) of (7) corresponds to the term \( X_1^{t_1} \cdot X_1^{t_2} \cdot \cdots \cdot X_1^{t_r} \) of (2).

**Proof:** It is easy to show by Definitions 4 and 5. Q.E.D.

**Example 8:** Consider the function \( f(X) \) of Example 5 and the function \( g(Y) \) of Example 7. For the term \( Y_1^{0/1} \cdot Y_2^{1/2} \cdot Y_3^{2/3} \) of \( g(Y) \), the corresponding term of \( f(X) \) is \( X_1^{0/1, 10} \cdot X_2^{0/1, 10} \cdot X_3^{0/1, 10} \). For the term \( Y_1^{0/1} \cdot Y_2^{1/2} \cdot Y_3^{2/3} \), the corresponding term of \( f(X) \) is

\[ X_1^{0/1, 10} \cdot X_2^{0/1, 10} \cdot X_3^{0/1, 10} \]

And for the term \( Y_1^{1/2} \), the corresponding term of \( f(X) \) is \( X_1^{0/1, 10} \cdot X_2^{0/1, 10} \).

**Theorem 3:** Let \( (X_1, X_2, \ldots, X_r) \) be a partition of \( X \). Suppose that \( f \) and \( g \) satisfy the relation

\[ f(\psi_1(X_1), \psi_2(X_2), \ldots, \psi_r(X_r)) = g(Y_1, Y_2, \ldots, Y_r) \]  

(9)
$f(X_1, X_2, \ldots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \ldots, \Psi_r(X_r))$.

If $P_1$ and $Q_1$ are minimal sum-of-products expressions for $f$ and $g$, respectively, then $t(P_1) = t(Q_1) \leq \left( \prod_{i=1}^r k_i \right) / \left( \max_{i} |k_i| \right)$, where $\Psi_i : B^{n_i} \rightarrow M_i ; M_i = \{0, 1, \ldots, k_i - 1\}, n_i = d(X_i)$, and $1 \leq |M_i| = k_i \leq 2^{n_i}$.

Proof: 1) For $P_1$, a minimal sum-of-products expression of $f(X)$

$$f(X_1, X_2, \ldots, X_r) = \bigvee_{(S_1, S_2, \ldots, S_r)} X_1^{S_1} \cdot X_2^{S_2} \cdots X_r^{S_r}$$

consider the expression $P_2$, which has the form

$$\bigvee_{(G_1, G_2, \ldots, G_r)} Y_1^{G_1} \cdot Y_2^{G_2} \cdots Y_r^{G_r}, \quad \text{where } G_i = \Psi_i^{-1}(S_i).$$

Clearly, $t(P_1) = t(P_2)$. It is easy to show that $P_2$ represents $g(Y)$. For $Q_1$, a minimal sum-of-products expression of $g(Y)$

$$g(Y_1, Y_2, \ldots, Y_r) = \bigvee_{(T_1, T_2, \ldots, T_r)} Y_1^{T_1} \cdot Y_2^{T_2} \cdots Y_r^{T_r}$$

consider the expression $Q_2$, which has the form

$$\bigvee_{(D_1, D_2, \ldots, D_r)} X_1^{D_1} \cdot X_2^{D_2} \cdots X_r^{D_r}, \quad \text{where } D_i = \Psi_i^{-1}(T_i).$$

Clearly, $t(Q_1) = t(Q_2)$. By Lemma 5, $Q_2$ represents $f(X)$. As $P_1$ is a minimal expression of $f(X)$, we have $t(P_1) \leq t(Q_1)$. As $Q_1$ is a minimal expression of $g(Y)$, we have $t(Q_1) \leq t(P_2)$. Therefore, $t(P_1) = t(Q_1)$.

2) Assume without loss of generality that $\max_{i} |k_i| = k_1$. $g(Y)$ can be represented in the form

$$g(Y_1, Y_2, \ldots, Y_r) = \bigvee_{(T_1, \ldots, T_r)} Y_1^{T_1} \cdot Y_2^{T_2} \cdots Y_r^{T_r} \quad (9)$$

where $T_1 \subseteq M_1$ and $T_i \subseteq M_i (i = 2, 3, \ldots, r)$. The number of terms in (9) is at most

$$\prod_{i=2}^r k_i = \left( \prod_{i=1}^r k_i \right) / \left( \max_{i} |k_i| \right).$$

Q.E.D.

Example 9: The expression (8) of Example 4 is a minimal sum-of-products expression for $g(Y)$. Therefore, the corresponding minimal expression for $f(X)$ is given by

$$f(X_1, X_2, X_3) = X_1^{00} \cdot X_2^{01} \cdot X_3^{00} \cdot X_3^{01} \cdot X_3^{10} \cdot X_3^{11} \vee X_1^{10} \cdot X_2^{11} \cdot X_3^{11} \vee X_1^{11} \cdot X_2^{11} \cdot X_3^{11}.$$  

Theorem 4: Let $(X_1, X_2, \ldots, X_r)$ be a partition of $X$. Suppose that $f$ can be represented in a form $f(X_1, X_2, \ldots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \ldots, \Psi_r(X_r))$, where $\Psi_i : B^{m_i} \rightarrow M_i ; M_i = \{0, 1, \ldots, k_i - 1\}; 1 \leq k_i = |M_i| \leq 2^{m_i}$ is a partition function. There exists a function $f(X_1, X_2, \ldots, X_r)$ whose minimal sum-of-products expression contains $\left( \prod_{i=1}^r k_i \right) / \left( \max_{i} |k_i| \right)$ terms.

Proof: Assume without loss of generality that $\max_{i} |k_i| = k_1$. Let $g(Y)$ be

$$g(Y_1, Y_2, \ldots, Y_r) = \begin{cases} 1 & \text{if } Y_1 + Y_2 + \ldots + Y_r = 0 \pmod{k_1} \\ 0 & \text{otherwise} \end{cases}$$

$g(Y)$ can be written as

$$g(Y_1, Y_2, \ldots, Y_r) = \bigvee_{t_1 + t_2 + \ldots + t_r = 0 \pmod{k_1}} Y_1^{t_1} \cdot Y_2^{t_2} \cdots Y_r^{t_r} \quad (10)$$

Every expression for $g(Y)$ has the form (10) because if the expression has a term of the form

$$Y_1^{t_1} \cdot Y_2^{t_2} \cdots Y_i^{t_i} \cdots Y_r^{t_r}, \quad |S_i| \geq 2$$

then it cannot satisfy the condition for $g(Y)$. Therefore, the minimal expression which represents $g(Y)$ has the form (10).

For arbitrary $t_2, t_3, \ldots, t_r$, there exists $t_1 \in \{0, 1, \ldots, k_1 - 1\}$ such that $t_1 + t_2 + \ldots + t_r = 0 \pmod{k_1}$. So the number of the terms of (10) is $\prod_{i=2}^r k_i$. By Theorem 3 the number of the terms of $f$ is also $\prod_{i=2}^r k_i$.

Q.E.D.

Theorem 5: Let $(X_1, X_2, \ldots, X_r)$ be a partition of $X, d(X_i) = q_i$ and $q_r = n$. The necessary and sufficient number of terms to represent an arbitrary function $f(X_1, X_2, \ldots, X_r)$ is

$$2^{n-q} - q$$

Proof—Sufficiency: By Theorem 2 the minimal sum-of-products expression for $f$ contains at most $2^{n-q}$ terms.

Necessity: Let the function $f(X)$ be decomposed as

$$f(X_1, X_2, \ldots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \ldots, \Psi_r(X_r))$$

where

$$\Psi_i(X_i) = \bigvee_{k=1}^{q_i} X_i^{2^{q_i-k}}, X_i = \{x_{i1}, x_{i2}, \ldots, x_{i1}\}$$

and let $\Psi_i$ denote a partition function $\Psi_i : B^{q_i} \rightarrow M_i ; M_i = \{0, 1, \ldots, 2^{q_i} - 1\}$. By Theorem 4 there exists a function whose minimal sum-of-products expression contains $(2^{q_i})^{r-1}$ terms.

Q.E.D.

Example 10: Consider the function $f_1$ in Table 6. Let $X_1 = (x_1, x_2)$ and $X_2 = (x_3, x_4), f_1$ can be decomposed as $f_1(X_1, X_2) = g_1(\Psi_1(X_1), \Psi_2(X_2))$, where $\Psi_i(00) = 0, \Psi_i(01) = 1, \Psi_i(10) = 2, \Psi_i(11) = 3, (i = 1, 2)$, and

$$g_1(Y_1, Y_2) = \begin{cases} 1 & Y_1 + Y_2 = 0 \pmod{4} \\ 0 & \text{otherwise} \end{cases}$$

The minimal sum-of-products expression for $g_1(Y_1, Y_2)$ is

$$g_1(Y_1, Y_2) = Y_1^0 \cdot Y_2^0 \vee Y_1^1 \cdot Y_2^1 \vee Y_1^2 \cdot Y_2^2 \vee Y_1^3 \cdot Y_2^3$$

and the corresponding expression for $f_1(X_1, X_2)$ is

$$f_1(X_1, X_2) = X_1^{00} \cdot X_2^{00} \vee X_1^{01} \cdot X_2^{01} \vee X_1^{10} \cdot X_2^{10} \vee X_1^{11} \cdot X_2^{11}.$$
Therefore, the minimal sum-of-products expression for \( f_1 \) contains \( 2^{n-q} = 4 \) terms. This is an example of a function of Theorem 5, where \( r = q = 2 \).

**Definition 6:** Let \((X_1, X_2, \ldots, X_r)\) be a partition of \( X \), \( f(X) \) is said to be partially symmetric with respect to \( X_i \) if \( f(X) \) is invariant under any permutation of variables in \(|X_i|\).

**Theorem 6:** Let \((X_1, X_2, \ldots, X_r)\) be a partition of \( X \), \( d(X_i) = q \) and \( qr = n \). Let \((X, \text{primary symmetric with respect to } X_i)\) for all \( i = 1, 2, \ldots, r \). The necessary and sufficient number of terms to represent \( f \) is \((q + 1)^{r-1}\).

**Proof—Sufficiency:** By the property of the partially symmetric function, \( f(X) \) can be decomposed as

\[
\begin{align*}
f(X_1, X_2, \ldots, X_r) &= g_1(\Psi_1(X_1), \\
\Psi_2(X_2), & \ldots, \Psi_r(X_r)) \tag{11}
\end{align*}
\]

where \( \Psi_i(X_i) = |X_i| \) \((0 \leq |X_i| \leq q, \ |X_i| \) denotes the number of 1's in \( X_i \)). By Theorem 3 the minimal sum-of-products expression for \( f \) contains \((q + 1)^{r-1}\) terms.

**Necessity:** By Theorem 4 there exists a function whose minimal sum-of-products expression contains \((q + 1)^{r-1}\) terms.

**Q.E.D.**

**Example 11:** Consider the function \( f_2 \) in Table VI. Let \( X_1 = (x_1, x_2) \) and \( X_2 = (x_3, x_4) \). \( f_2 \) is partially symmetric with respect to \( X_1 \), and it can be decomposed as

\[
f_2(X_1, X_2) = g_2(\Psi_1(X_1), \Psi_2(X_2)),
\]

where \( \Psi_1(00) = 0, \Psi_1(01) = \Psi_1(10) = 1, \Psi_1(11) = 2, (i = 1, 2) \) and

\[
g_2(Y_1, Y_2) = \begin{cases} 1 & \text{if } Y_1 + Y_2 = 0 \text{ (mod 3)} \\ 0 & \text{otherwise} \end{cases}
\]

The minimal sum-of-products expression for \( f_2 \) is

\[
g_2(Y_1, Y_2) = Y_1^2 \cdot Y_1 \lor Y_2 \cdot Y_2^2 \lor Y_1 \cdot Y_2^2
\]

and the corresponding expression for \( f_2 \) is

\[
f_2(X_1, X_2) = X_1^{00 \cdot 00} \cdot X_1^{01 \cdot 01} \cdot X_2^{01 \cdot 10} \cdot X_2^{01 \cdot 11} \cdot X_1^{11 \cdot 11} \cdot X_2^{01 \cdot 10}.
\]

Therefore, the minimal sum-of-products expression for \( f_2 \) contains \((q + 1)^{r-1} = 3 \) terms. This is an example of a function of Theorem 6, where \( q = r = 2 \).

**Theorem 7:** Let \((X_1, X_2, \ldots, X_r)\) be a partition of \( X \), and \( d(X_i) = q \). Let \( f(X) \) be a parity function of \( n \) variables. The number of terms in the minimal sum-of-products expression for \( f \) is \( 2^{r-1} \).

**Proof:** By the property of a parity function, \( f(X) \) can be decomposed as \( f(X_1, X_2, \ldots, X_r) = g(\Psi_1(X_1), \Psi_2(X_2), \ldots, \Psi_r(X_r)) \), where

\[
\begin{align*}
\Psi_i(X_i) &= \begin{cases} 1 & \text{if } |X_i| = 0 \text{ (mod 2)} \\ 0 & \text{if } |X_i| = 0 \text{ (mod 2)} \end{cases}
\end{align*}
\]

By Theorem 3 the minimal sum-of-products expression of \( f \) contains at most \( 2^{r-1} \) terms. Clearly, \( \Psi_i : [0, 1] \rightarrow \{0, 1\} \) is a partition function. By Theorem 4 we need \( 2^{r-1} \) terms to represent \( f \).

**Q.E.D.**

**Example 12:** Consider the function \( f_3 \) in Table VI. Let \( X_1 = (x_1, x_2) \) and \( X_2 = (x_3, x_4) \). \( f_3 \) is parity function and it can be decomposed as \( f_3(X_1, X_2) = g_3(\Psi_1(X_1), \Psi_2(X_2)) \), where \( \Psi_1(00) = \Psi_1(11) = 0, \Psi_1(01) = \Psi_1(10) = 1, \) and

\[
g_3(Y_1, Y_2) = \begin{cases} 1 & \text{if } Y_1 + Y_2 = 0 \text{ (mod 2)} \\ 0 & \text{if } Y_1 + Y_2 = 1 \text{ (mod 2)} \end{cases}
\]

The minimal sum-of-products expression for \( g_3(Y_1, Y_2) \) is

\[
g_3(Y_1, Y_2) = Y_1 \lor Y_2 \lor Y_1 \lor Y_2
\]

and the corresponding expression for \( f_3(X_1, X_2) \) is

\[
f_3(X_1, X_2) = X_1^{00 \cdot 00} \cdot X_1^{01 \cdot 01} \cdot X_2^{01 \cdot 11} \cdot X_2^{01 \cdot 10}.
\]

Therefore, the minimal sum-of-products expression for \( f_3 \) contains \( 2^{r-1} = 2 \) terms. This is an example of a function of Theorem 7, where \( q = r = 2 \).

**IV. COMPLEXITY OF PROGRAMMABLE LOGIC ARRAYS WITH DECODERS**

In Section IV-A we show that a PLA with decoders realizes a generalized Boolean function. In Section IV-B we obtain the sizes of PLA's for various classes of functions.

**A. Realization of Functions Using PLA's with Decoders**

**Definition 7:** Let \( a = (a_1, a_2, \ldots, a_n) \) be a constant in \( B^n \). \( X^n \) is said to be a maxterm of \( X \).

By Definition 2 we have the following.

**Lemma 6:** \( X^S \) can be represented by the product of some maxterms of \( X \)

\[
X^S = \bigwedge_{a \in (B^n - S)} \overline{X^a}.
\]

**Theorem 8:** Let \((X_1, X_2, \ldots, X_r)\) be a partition of \( X \). In a PLA with decoders, if each decoder generates all the maxterms of \( X_i \), for \( i = 1, 2, \ldots, r \), then an arbitrary term which has the form \( X^S \cdot X^S \cdot \cdots X^S \) can be realized in each column of the AND array. The width \( W \) of the AND array to realize the given function \( f \) is equal to the number of terms in a minimal sum-of-products expression for \( f \) and \( W \leq 2^n - \text{max} |S| \).

**Example 13:** In the Case of the Two-Level PLA. Let \( X_1 = (x_1), X_2 = (x_2), X_3 = (x_3), \) and \( X_4 = (x_4) \) be a (trivial) partition of \( X = (x_1, x_2, x_3, x_4) \). Fig. 2 shows the two-level PLA for the function of Table I. Each column of the AND array corresponds to each term of (4).

**b) In the Case of the PLA with Two-Bit Decoders:** Let \( X = (X_1, X_2), X_1 = (x_1, x_2), \) and \( X_2 = (x_3, x_4) \) be a partition of \( X = (x_1, x_2, x_3, x_4) \). Fig. 4 shows the PLA with two-bit de-
coders for the function of Table 1. Each column of the AND array corresponds to each term of (5). (End of the example.)

By Theorem 8, to minimize the size of the AND array for \( f(X) \), it is sufficient to obtain a minimal sum-of-products expression of \( f(X) \) having the form (2). In the case of a two-level PLA, the expression denotes the function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), whereas in the case of a PLA with two-bit decoders, the expression denotes the two-valued function of four-valued variables \( f : \{00, 01, 10, 11\}^{n/2} \rightarrow \{0, 1\} \).

**Theorem 9:** Let \( W_1 \) and \( W_2 \) be the width of the two-level PLA and the PLA with two-bit decoders to realize a function, respectively. Then \( W_1 \leq W_2 \).

**B. Formulas for PLA’s with Decoders**

**Theorem 10:** In a PLA with \( q \)-bit decoders the following size is necessary and sufficient to realize an \( n \)-variable function, when the assignment of the input variables is fixed to the decoders where \( n = qr \).

1) For an Arbitrary Function: \( (r \cdot 2^q + 1) \cdot 2^r \).
2) For an Arbitrary Symmetric Function: \( (r \cdot 2^q + 1) \cdot (q + 1)^{-1} \).
3) For a Parity Function: \( (r \cdot 2^q + 1) \cdot 2^{-1} \).

**Proof:** By Theorems 5-7. Q.E.D.

**Corollary 1:** In a PLA with two-bit decoders the following size is necessary and sufficient to realize an \( n \)-variable function, when the assignment of the input variables is fixed to the decoders \( (n = 2r) \).

1) For an Arbitrary Function: \( \frac{1}{2}(n + \frac{1}{2}) \cdot 2^n \).
2) For an Arbitrary Symmetric Function: \( \frac{3}{2}(n + \frac{1}{2}) \cdot 3^n \).
3) For a Parity Function: \( (n + \frac{1}{2}) \cdot 2^n \).

**Corollary 2:** The size of two-level PLA to realize a parity function of \( n \) variable is \( (n + \frac{1}{2}) \cdot 2^n \).

**C. Assignment Problem of the Input Variables to the Decoders**

In the case of the PLA with decoders the way of assignment of the input variables to the decoders often influences the size of the PLA.

**Example 14:** Let us realize the function of Table 1 by using a PLA with two-bit decoders. Assume that \( X = (X_1, X_2) \) is a partition of the input variables \( X \). There exist three possible ways of assignment of four input variables to two two-bit decoders.

1) When the input variables are assigned as \( X_1 = (x_1, x_2) \) and \( X_2 = (x_3, x_4) \). (See Fig. 4.) The minimal sum-of-products expression is

\[
\begin{align*}
\text{f}(X_1, X_2) &= X_1^{00,01} \cdot X_2^{00,01} \vee X_1^{00,11} \cdot X_2^{01,10} \vee X_1^{01,10} \cdot X_2^{00,11}.
\end{align*}
\]

So three columns are necessary in this assignment.

2) When the input variables are assigned as \( X_1 = (x_1, x_3) \) and \( X_2 = (x_2, x_4) \). (See Fig. 7.) The minimal sum-of-products expression is

\[
\begin{align*}
\text{f}(X_1, X_2) &= X_1^{00,10} \cdot X_2^{01,11} \vee X_1^{01,11} \cdot X_2^{00,11}.
\end{align*}
\]

So two columns are necessary in this assignment.

3) When the input variables are assigned as \( X_1 = (x_1, x_4) \) and \( X_2 = (x_2, x_3) \). (See Fig. 8.) The minimal sum-of-products expression is

\[
\begin{align*}
\text{f}(X_1, X_2) &= X_1^{00,01} \cdot X_2^{01,10} \vee X_1^{01,10} \cdot X_2^{00,10} \vee X_1^{01,11} \cdot X_2^{00,11} \cdot X_1^{00,11} \cdot X_2^{00,11}.
\end{align*}
\]

So three columns are necessary in this assignment.

Therefore, when the input variables are assigned as shown in Fig. 7, the array is the minimum.

**Example 15:** Consider the function \( f_1 \) in Table VI. When the input variables are assigned as \( X_1 = (x_1, x_3) \) and \( X_2 = (x_2, x_4) \), \( f_1(X_1, X_2) \) can be written as

\[
\begin{align*}
f_1(X_1, X_2) &= X_1^{00} \cdot X_2^{00} \vee X_1^{01} \cdot X_2^{11} \vee X_1^{11} \cdot X_2^{00} \cdot X_1^{11} \cdot X_2^{11}.
\end{align*}
\]

Therefore, the function \( f_1 \) requires only two terms in this assignment.

**D. Statistical Results**

**Average Size of PLA’s with Two-Bit Decoders:** Table III shows the average size of PLA’s for up to 10-variable functions. \( d = (u/2^n) \times 100 \) denotes the percentage of minterms which are mapped to one.

**The Effect of the Assignment of Input Variables to the Decoders:** To investigate the dependence of the way of assignment of input variables, ten functions of 8 variables were randomly generated for each density. Then 105 expressions which correspond to all possible ways of assignments were minimized. Table VII shows the statistical result of this exhaustive investigation.
TABLE VII
AVERAGE NUMBER OF COLUMNS OF PLA’S FOR EIGHT-VARIABLE FUNCTIONS

<table>
<thead>
<tr>
<th>Type of PLA</th>
<th>5%</th>
<th>10%</th>
<th>15%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLA’s with two-bit decoders</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Assignment is optimal</td>
<td>1.68</td>
<td>1.97</td>
<td>1.94</td>
<td>1.85</td>
<td>1.76</td>
<td>1.65</td>
</tr>
<tr>
<td>Assignment is non-optimal</td>
<td>1.91</td>
<td>2.20</td>
<td>2.17</td>
<td>2.10</td>
<td>1.99</td>
<td>1.90</td>
</tr>
</tbody>
</table>

Each entry is the average of 10 randomly generated functions.
* The average of 100 assignments.

Multiple-Output Function: In the case of m-output function

\[ f_j : B^n \times B^n \times \cdots \times B^n \rightarrow B \quad (j = 0, 1, \cdots, m - 1) \]

the number of columns of the AND array is equal to the number of the terms of the expression for the function \( f_j \).

\[ f_j : B^n \times B^n \times \cdots \times B^n \times M \rightarrow B, \]

where \( M = \{0, 1, \cdots, m - 1\} \).

Table VIII shows the average number of columns of the AND arrays for n-input 4-output functions, where \( d = \lfloor \log_2 (2^n \cdot m) \rfloor \) and \( u = \lfloor \log_2 (\log_2 (m + 1)) \rfloor \).

V. COMPLEXITY OF THREE-LEVEL PROGRAMMABLE LOGIC ARRAYS

In this section, first we show that a three-level PLA realizes a generalized Boolean function. Then we obtain the sizes of PLA’s for various classes of functions.

Lemma 7: Let \( (X_1, X_2, \cdots, X_r) \) be a partition of \( X \). In a three-level PLA, if the D array generates all the maxterms of \( X_i \) for every \( i \) (\( i = 1, 2, \cdots, r \)), then an arbitrary function which has a form \( X_1^{S_1} \cdot X_2^{S_2} \cdot \cdots \cdot X_r^{S_r} \) can be generated in each column of the AND array. The width \( W \) of the AND array to realize the given function \( f \) is equal to the number of terms in a minimal sum-of-products expression for \( f \), and \( W \leq 2^{n - \max|n_i|} \).

By Lemma 7 we have the following.

Theorem 11: Let \( (X_1, X_2, \cdots, X_r) \) be a partition of \( X \) and the D array generate all the maxterms of \( X_i \) for every \( i \). The size of three-level PLA which is sufficient to realize an arbitrary \( n \)-variable function is given by the following formula:

\[ C(n) = (2^n + W)H + W \] (12)

where

\[ W = 2^{n - \max|n_i|}, \quad H = \sum_{i=1}^{r} 2^{n_i}, \quad \text{and} \quad n_i = d(X_i). \]

Definition 8: Let \( (X_1, X_2, \cdots, X_r) \) be a partition of \( X \). A vector \( n = (n_1, n_2, \cdots, n_r) \) is said to be a partition vector of the input variables, where \( n_i = d(X_i) \).

Let \( n = \left[ \left\lfloor \frac{n}{2} \right\rfloor, 1, 1, \cdots, 1 \right] \), where \( \left\lfloor \frac{n}{2} \right\rfloor \) denotes the largest integer not exceeding \( \frac{n}{2} \), we have the following.

Theorem 12: The size of three-level PLA which is sufficient to realize an arbitrary function of \( n \)-variable is given by the following formulas:

\[ 2^n + (3n + 1)2^{\frac{n}{2}} + 2n^2, \quad \text{when} \ n \ \text{is even}. \]

\[ 2^n + (2n + 2)2^{\frac{n}{2} + 1} + 2n(n + 1), \quad \text{when} \ n \ \text{is odd}. \]

By Theorem 3 we have the following.

Theorem 13: Let \( (X_1, X_2, \cdots, X_r) \) be a partition of \( X \), and \( f(X) \) be partially symmetric with respect to \( X_i \) for all \( i \) (\( i = 1, 2, \cdots, r \)). The size of three-level PLA which is sufficient to realize \( f(X) \) is given by

\[ C(n) = (2^n + W)H + W \]

where

\[ W = \prod_{i=1}^{r} \left( n_i + 1 \right) \left[ \max \left\{ n_i + 1 \right\} \right], \]

\[ H = \sum_{i=1}^{r} 2^{n_i}, \quad \text{and} \quad n = (n_1, n_2, \cdots, n_r) \]

is a partition vector.

By Theorem 7 we have the following.

Theorem 14: The size of three-level PLA which is sufficient to realize an \( n \)-variable parity function is given by

\[ C(n) = (2^n + W)H + W \]

where

\[ W = 2^{r-1} \quad \text{and} \quad H = \sum_{i=1}^{r} 2^{n_i}. \]

Table IX shows the value of \( C(n) \) calculated from the formulas (12)–(14) and corresponding partition vector \( n \), where \( n \) is chosen to minimize the value of \( C(n) \).

VI. CONCLUSIONS

1) Generalized Boolean functions can be directly realized by PLA’s with decoders or three-level PLA’s.

2) In a PLA with two-bit decoders, the following size is necessary and sufficient to realize an \( n \)-variable function when the assignment of the input variables to the decoders is fixed:
for an arbitrary function, $\frac{1}{2}(n + \frac{1}{2})2^n$; for an arbitrary symmetric function, $\frac{3}{2}(n + \frac{1}{2})\sqrt{3^n}$; for a parity function, $(n + \frac{1}{2})\sqrt{2^n}$.

3) A PLA with two-bit decoders requires a smaller array than a two-level PLA. In the case of $n = 10$ and $d = 50$ percent the former is, on the average, 24 percent smaller than the latter.

4) The size of the arrays of PLAs with two-bit decoders can be reduced by optimizing the assignment of the input variables. In the case of $n = 8$ and $d = 40$ percent optimally assigned PLAs' are 10 percent smaller than nonoptimally assigned ones.

5) For a three-level PLA, the following size is sufficient to realize an $n$-variable function ($n = 2r$): $2^n + (3n + 1)\sqrt{2^n} + 2n^2$.

ACKNOWLEDGMENT

The author would like to thank Prof. H. Ozaki, Prof. H. Terada, Prof. K. Kinoshita, and Dr. H. Fujiwara for their encouragement. The author is grateful to Mr. Ishikawa for programming. He would also like to thank the referees for their careful reading of the manuscript.

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